The pullback theorem for (relative) monads

Nathanael Arkor
Tallinn University of Technology

Dylan McDermott
Reykjavik University

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Idea

By working with relative monads, we may better understand classical results about (non-relative) monads.

The pullback theorem

Algebras and free algebras

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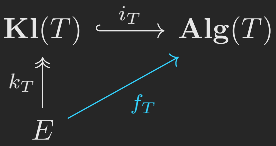

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$$\begin{array}{ccc} \mathbf{Kl}(T) & \xhookrightarrow{\;i_T\;} & \mathbf{Alg}(T) \\ {\scriptstyle k_T}\big\uparrow & \nearrow_{f_T} & \\ E & & \end{array}$$

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We call this characterisation the pullback theorem for monads.

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In fact, for many monads T, the pullback theorem can be refined to obtain a sharper characterisation of the T-algebras in terms of a pullback over a nerve functor, rather than the Yoneda embedding.

Definition 2

Let $j \colon A \to E$ be a functor. The nerve of j is the restricted Yoneda embedding $n_j \colon E \to \widehat{A}$ defined by:

$$n_j(e) := E(j-, e)$$

The nerve of the identity 1_E is the Yoneda embedding $よ_E \colon E \to \widehat{E}$.

Nerve theorems for monads II

A nerve theorem for a class of monads takes the following shape.

Theorem schema

Let T be a nice monad on a category E. Then there exists a full subcategory $j \colon A \hookrightarrow E$ such that, denoting by $k \colon A \to \mathbf{Kl}_A(T)$ the full subcategory of $\mathbf{Kl}(T)$ spanned by the objects of A, the following diagram exhibits a pullback in $\mathbb{C}\mathbf{at}$.

$$\begin{array}{ccc} \mathbf{Alg}(T) & \hookrightarrow & \widehat{\mathbf{Kl}_A(T)} \\ {\scriptstyle u_T}\downarrow & \lrcorner & \downarrow{\scriptstyle \widehat{k}} \\ E & \xhookrightarrow[n_j]{} & \widehat{A} \end{array}$$

Example A: algebraic theories

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Then $k \colon \mathbb{F} \to \mathbf{Kl}_{\mathbb{F}}(T)$ is the finitary algebraic theory associated to T, and the category of T-algebras forms a pullback over n_j, exhibiting it as the category of models for the algebraic theory k [Lin69; Die74].

$$\begin{array}{ccccc}
\mathbf{Cart}(\mathbf{Kl}_{\mathbb{F}}(T)^{\mathrm{op}}, \mathbf{Set}) & \xrightarrow{\simeq} & \mathbf{Alg}(T) & \hookrightarrow & \widehat{\mathbf{Kl}_{\mathbb{F}}(T)} \\
{\scriptstyle \mathbf{Cart}(k^{\mathrm{op}}, \mathbf{Set})}\downarrow & & \downarrow {\scriptstyle u_T} \quad \lrcorner & & \downarrow {\scriptstyle \widehat{k}} \\
\mathbf{Cart}(\mathbb{F}^{\mathrm{op}}, \mathbf{Set}) & \xrightarrow[\simeq]{} & \mathbf{Set} & \hookrightarrow & \widehat{\mathbb{F}}
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Example B: categories

Let T be the free category monad on $\mathbf{Grph}$ and take $\Delta_0 \hookrightarrow \mathbf{Grph}$ to be the full subcategory of linear graphs.

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$$\begin{array}{ccc} \mathbf{Cat} & \hookrightarrow & \widehat{\Delta} \\ {\scriptstyle u_T}\downarrow & \lrcorner & \downarrow{\scriptstyle \widehat{k}} \\ \mathbf{Grph} & \hookrightarrow & \widehat{\Delta_0} \end{array}$$

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An similar characterisation holds for the free strict ω-category monad on the category of globular sets [Ber02].

Frameworks for nerve theorems

Nerve theorems have been established for several different classes of monads.

- *Familially representable monads* [Lei04; Web07]
- *Monads with arities* [Web07; Mel10; BMW12]
- Φ*-cocontinuous monads* [NP09; LR11; Luc16]
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Consequently, the natural question to ask is: *of what general phenomenon are these theorems instances?*

Relative monads

Relative monads

A relative monad is a generalisation of a monad, where the underlying functor is permitted to be an arbitrary functor, rather than an endofunctor.

Definition 3 ([ACU10])

A *relative monad* comprises

1. a functor $j \colon A \to E$, the *root*;
2. a functor $t \colon A \to E$, the *carrier*;
3. a natural transformation $\eta \colon j \Rightarrow t$, the *unit*;
4. a natural transformation $\dagger \colon E(j-, t-) \Rightarrow E(t-, t-)$, the *extension operator*,

satisfying unitality and associativity axioms.

When $j = 1$, this is equivalent to the usual definition of a monad.

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The Kleisli category for relative monad

Just as for non-relative monads, there are two important categories associated to every relative monad.

Definition 4 ([ACU10])

Let $j\colon A \to E$ be a functor and let T be a j-relative monad. The Kleisli category for T is the category $\mathbf{Kl}(T)$ defined by

$$|\mathbf{Kl}(T)| := |A|$$
$$\mathbf{Kl}(T)(x, y) := E(jx, ty)$$

with identities and composition given as in the Kleisli category for a monad.

This is equipped with an inclusion functor $k_T\colon A \to \mathbf{Kl}(T)$.

The category of algebras for relative monad

Definition 5 ([ACU10])

Let $j\colon A \to E$ be a functor and let T be a j-relative monad. A T-*algebra* comprises

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The category of T-algebras is the category $\mathbf{Alg}(T)$ whose objects are T-algebras and whose morphisms are morphisms in E preserving the algebra structure.

This is equipped with a forgetful functor $u_T\colon \mathbf{Alg}(T) \to E$.

The pullback theorem for relative monads

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Theorem 6

Let $j \colon A \to E$ be a dense functor and let T be a j-relative monad. The following diagram exhibits a pullback in $\mathbb{C}\mathbf{at}$.

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How does this relate to nerve theorems for monads?

Relative monads from monads

Let $T = (t, \mu, \eta)$ be a monad on a category E and let $j \colon A \to E$ be a functor. Precomposing j induces a j-relative monad structure on $(j \mathbin{;} t)$, which we denote by $(j \mathbin{;} T)$.

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We can characterise the category of free $(j \mathbin{;} T)$-algebras in terms of the category of free T-algebras: it is given by taking the full image factorisation of $A \to \mathbf{Kl}(T)$.

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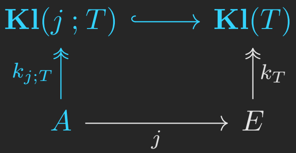

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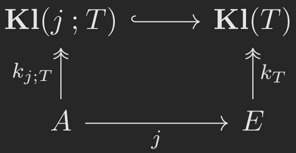

In particular, when $j \colon A \to E$ is the inclusion of a full subcategory, $\mathbf{Kl}(j \mathbin{;} T)$ is precisely the category $\mathbf{Kl}_A(T)$.

j-ary monads

In general, we cannot characterise the $(j\,;T)$-algebras in terms of the T-algebras. However, we do always have a canonical comparison functor, as follows.

$$\begin{array}{ccc} \mathbf{Alg}(T) & \dashrightarrow & \mathbf{Alg}(j\,;T) \\ & {\scriptstyle u_T} \searrow \quad \swarrow {\scriptstyle u_{j;T}} & \\ & E & \end{array}$$

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Theorem 8 ([AM23])

T is j-ary $\iff$ $u_{j;T} \colon \mathbf{Alg}(j\,;T) \to E$ admits a left adjoint.

The nerve theorem for j-ary monads

Theorem 9

Let $j\colon A \to E$ be a dense functor and let T be a monad on E. The following diagram exhibits a pullback in $\mathbb{C}\mathbf{at}$ if and only if T is j-ary.

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Towards a formal understanding

Formal category theory

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For instance, the nerve theorem holds for categories enriched in any monoidal category $\mathbb{V}$ (with no additional assumptions).

The pullback theorem, formally

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It is relatively straightforward to write out a direct proof of the pullback theorem. It is, however, rather fiddly. Furthermore, this proof strategy becomes less tenable once we move beyond ordinary categories to the formal setting.

This motivates the search for a more abstract approach to the pullback theorem.

The virtual equipment of categories

Categories, functors ($\to$), distributors ($\nrightarrow$), and natural transformations ($\Rightarrow$) form a virtual double category $\mathbb{C}\mathbf{at}$. In fact, $\mathbb{C}\mathbf{at}$ is a particularly well behaved kind of virtual double category called a virtual equipment.

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To reformulate the pullback theorem in the setting of a virtual equipment, we must introduce one more definition.

Semanticisers

A semanticiser is a kind of double categorical limit that resembles a pullback.

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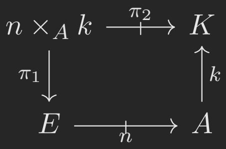

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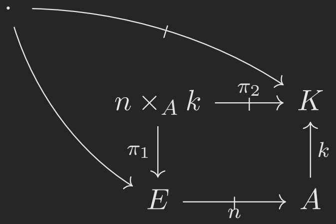

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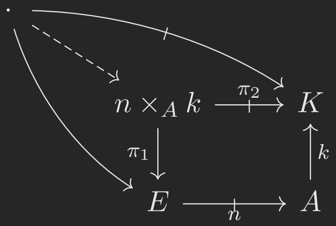

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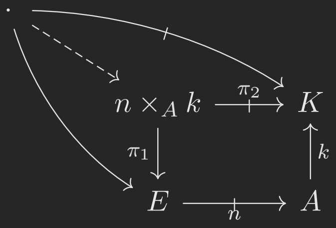

The concept of semanticiser makes sense in any (virtual) equipment.

The semanticiser theorem

With the notion of semanticiser at hand, we may reformulate the pullback theorem by eliminating the use of presheaf categories.

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Theorem 10

Let $j \colon A \to E$ be a dense functor and let T be a j-relative monad. The following diagram exhibits a semanticiser in $\mathbb{C}\mathbf{at}$.

$$\begin{array}{ccc} \mathbf{Alg}(T) & \longrightarrow & \mathbf{Kl}(T) \\ {\scriptstyle u_T}\big\downarrow & & \big\uparrow{\scriptstyle k_T} \\ E & \xrightarrow[E(j,1)]{\qquad} & A \end{array}$$

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The reason for this stems from a deep relationship between relative monads, loose-monads (a.k.a. *promonads*), and two-dimensional exactness.

Semanticisers from pullbacks

Semanticisers can be constructed from pullbacks and presheaf objects. This allows us to reformulate a universal property involving distributors into one involving only functors, hence giving a more concrete description.

Theorem 11

Let $j \colon A \to E$ be a functor with small domain. The semanticiser of $E(j, 1) \colon E \nrightarrow A$ and $A \xrightarrow{k} K$ is exhibited by the following pullback.

$$\begin{array}{ccc} n_j \times_{\widehat{A}} \widehat{k} & \xrightarrow{\pi_1} & \widehat{K} \\ {\scriptstyle \pi_2}\downarrow & \lrcorner & \downarrow{\scriptstyle \widehat{k}} \\ E & \xrightarrow[n_j]{} & \widehat{A} \end{array}$$

The nerve theorem for relative monads

Thus, the pullback theorem follows as a consequence of the more general semanticiser theorem.

Corollary 12

Let $j \colon A \to E$ be a dense functor and let T be a j-relative monad. The following diagram exhibits a pullback in $\mathbb{C}\mathbf{at}$.

$$\begin{array}{ccc} \mathbf{Alg}(T) & \hookrightarrow & \widehat{\mathbf{Kl}(T)} \\ {\scriptstyle u_T}\downarrow \lrcorner & & \downarrow{\scriptstyle \widehat{k_T}} \\ E & \xrightarrow[n_j]{} & \widehat{A} \end{array}$$

Summary

- For a dense functor j, every j-relative monad T admits a pullback theorem, exhibiting the category of T-algebras as a pullback along the nerve of j.
- Prior nerve theorems for monads ([Lin69; Lei04; Web07; NP09; Mel10; LR11; BMW12; BG19; LP23]) arise as instances of the nerve theorem for j-ary monads.
- The pullback theorem follows as a consequence of the more general semanticiser theorem, which holds in any exact virtual equipment.

You can read our preprint here:

The pullback theorem for relative monads [AM24]

Do come and talk to me if you find any of this interesting!

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